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On surface waves with zero contact angle

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The linear, inviscid reflection of a straight-crested surface wave from a vertical wall is determined on the hypothesis that the contact angle of the meniscus vanishes. The reflection coefficient is a function of the parameter $\lambda \equiv k_0 l$, where k_0 is the wavenumber of the incident wave and l is the capillary length, and is approximated by $R = \exp(-4\mathrm{i}\lambda^2)$ for a gravity–capillary wave for which $\lambda \ll 1$. The solution of this reflection problem is used to obtain matched-asymptotic approximations for standing waves in channels and circular cylinders. The meniscus-induced, fractional reduction of the frequency of the dominant mode in a deep circular cylinder is $0.77\,\lambda^2$ (which exceeds the increase of $\frac{1}{2}\lambda^2$ associated with the capillary energy of the free surface). This decrement is within 2 mHz of the value inferred from the measurements of Cocciaro *et al.* (1991) after allowing for the reduction in frequency induced by the viscous boundary layers at the walls, but there are residual uncertainties (in this comparison) associated with the wetting process at the moving contact line and possible surface contamination.

1. Introduction

In 1876, Lord Rayleigh measured the periods of the first five standing-wave modes in a circular tank about 3 m in diameter by 'dipping one or more buckets synchronously with the beat of a metronome' and counting the oscillations for some 5 min after the withdrawal of the buckets; his results were within 1% of his theoretical predictions. Some 80 years later, Case & Parkinson (1957), working with electronic instrumentation, found discrepancies as large as 9% between theory and observation for the period of the dominant mode in a 'carefully polished' 3 in. cylinder and attributed them to 'surface tension effects associated with wetting of the wall'. This striking illustration of the importance of scale and, in particular, contact-line phenomena in the laboratory measurement of bounded surface waves is amplified by measurements of damping (see Miles 1967 for a review), the theoretical and experimental work of Benjamin & Scott (1979) on the fixed contact-line problem, the theoretical investigations of Hocking (1987*a*, *b*) and Miles (1990, 1991) using a general linear contact-line condition, and the experimental work of Cocciaro, Faetti & Nobili (1991) on the zero-contact-angle problem. The limiting cases of either fixed contact line or fixed contact angle (which is realistic only for zero contact angle) are of special experimental interest by virtue of the presumed absence of contact-line dissipation (but see remarks at end of §5).

As a basic zero-contact-angle problem, I consider here the linear, inviscid reflection of a surface wave of asymptotic form

$$y - y_{\mathrm{m}}(x) = \eta(x,t) \sim \mathrm{Re}\{A_{\mathrm{i}}\,\mathrm{e}^{\mathrm{i}\omega t}(\mathrm{e}^{\mathrm{i}k_0 x} + R\,\mathrm{e}^{-\mathrm{i}k_0 x})\} \quad (x \uparrow \infty) \tag{1.1}$$

by a vertical wall at $x = 0$ at which

$$\mathrm{d}y_{\mathrm{m}}/\mathrm{d}x = \infty \quad (x = 0), \tag{1.2}$$

where: $y = y_{\mathrm{m}}(x)$ is the static meniscus; η is the free-surface displacement; Re signifies *the real part of*; A_{i}, ω and k_0 are the complex amplitude, frequency and wavenumber of the incident wave; ω and k_0 are related by the linear dispersion relation

$$\omega^2 = gk_0 + Tk_0^3 \equiv gk_0(1+k_0^2 l^2), \tag{1.3}$$

T is the kinematic surface tension, and l is the capillary length; R is the reflection coefficient, the determination of which is the primary goal of the present investigation. This reflection problem is solved by Hocking (1987a) for the general contact-line condition $c\eta_x = \eta_t$, where c is a constant, but he neglects the meniscus.

I proceed as follows. In §2, I recapitulate the known solution for the static meniscus and formulate the linearized (in η) boundary-value problem for two-dimensional, irrotational motion with the dynamical boundary conditions projected on $y = y_{\mathrm{m}}$. In §3, I transform this problem to a pair of simultaneous integral equations for the Fourier-cosine transforms of η and the velocity potential. In §4, I obtain a first-order (in y_{m}) solution to these integral equations and show that the reflection coefficient is approximated by

$$R = \exp(-4\mathrm{i}\lambda^2), \quad \lambda \equiv k_0 l \ll 1. \tag{1.4a,b}$$

In §5, I extend the solution of the reflection problem to standing waves in a deep channel of width $b \gg l$ and show that the natural frequencies are approximated by

$$\omega_n^2 = (g/b)\,[n\pi + (n\pi-4)\,\lambda_n^2], \quad \lambda_n \equiv n\pi l/b \quad (n = 1,2,\ldots). \tag{1.5a,b}$$

Note that the meniscus reduces the resonant frequency of the dominant mode ($n = 1$) by an amount that exceeds the increase associated with the capillary energy of the free surface.

In §6, I determine the effects of the meniscus on standing waves in a circular cylinder of radius a and depth h ($a, h \gg l$) and show that the resonant frequencies are approximated by

$$\omega^2 = \kappa(g/a)\tanh(\kappa h/a)\,[1+\lambda^2(1-2\mu)], \tag{1.6a}$$

where

$$\lambda = \kappa l/a, \quad \mu = \kappa(\kappa^2 - m^2)^{-1}[1 + 2\kappa(h/a)\,\mathrm{cosech}\,(2\kappa h/a)], \tag{1.6b,c}$$

and κ is a zero of $J'_m(\kappa)$. The meniscus-induced, fractional decrement of ω for the dominant mode ($m = 1$, $\kappa = 1.84$) in deep water ($h > 2a$) is $0.77\lambda^2$, which, just as for the channel, exceeds the increase of $\frac{1}{2}\lambda^2$ associated with the capillary energy of the free surface.

The approximation (1.6) agrees with the frequency measured by Cocciaro *et al.* (1991) within 2 mHz after allowing for the reduction in frequency induced by the viscous boundary layers on the lateral wall and the bottom, but neglecting viscoelastic effects at the free surface. This neglect appears to be appropriate for their fluid ('octane'), but their measured damping exceeds the corresponding theoretical value. This excess damping could be due to surface contamination or the wetting process at the moving contact line.

The constraint of zero contact angle implies that no work is done by the capillary force at the contact line. There is a meniscus-induced change in the viscous dissipation at the wall, but the corresponding correction factor for this dissipation in the absence of the meniscus (see the Appendix) is $1 + O(\lambda^2)$, which is negligible in the present context.

2. Linear boundary-value problem

The meniscus, $y = y_m(x)$, is governed by the capillary equation

$$y = \frac{T}{g\mathscr{R}} = \frac{l^2 y''}{(1+y'^2)^{\frac{3}{2}}} \quad \left(y' \equiv \frac{\mathrm{d}y}{\mathrm{d}x}\right), \tag{2.1}$$

where $\mathscr{R}$ is the static radius of curvature. The solution of (2.1), subject to the conditions of zero contact angle ($y' = \infty$) at $x = 0$ and evanescence at $x = \infty$, has the parametric representation

$$\frac{x}{l} \equiv X = \log\left(\frac{1-\surd 2}{\tan\frac{1}{4}\chi}\right) + \surd 2 - 2\cos\tfrac{1}{2}\chi, \quad \frac{y_m}{l} \equiv Y_m = -2\sin\tfrac{1}{2}\chi, \tag{2.2 a, b}$$

where $\chi \equiv \tan^{-1} y'_m(x)$ increases monotonically from $-\frac{1}{2}\pi$ at $x = 0$ to 0 at $x = \infty$. We note that

$$\int_0^\infty Y_m \, \mathrm{d}X = 1. \tag{2.3}$$

The assumption of two-dimensional, incompressible, irrotational motion described by the velocity potential $\phi(x, y, t)$ and the free-surface displacement (relative to $y = y_m$) $\eta(x, t)$ leads to the linear (in ϕ and η) boundary-value problem

$$\phi_{xx} + \phi_{yy} = 0 \quad (0 < x < \infty, -\infty < y < y_m + \eta), \tag{2.4}$$

$$\phi_x = 0 \quad (x = 0), \quad \phi \to 0 \quad (y \downarrow -\infty), \tag{2.5 a, b}$$

$$\phi_y = \eta_t + y'_m \phi_x, \quad \phi_t + g\eta = T[(1+y'^2_m)^{-\frac{3}{2}} \eta_x]_x \quad (y = y_m). \tag{2.6 a, b}$$

Note that $(1+y'^2_m)^{-\frac{3}{2}} = \cos^3\chi$ vanishes like $x^{\frac{3}{2}}$ as $x \downarrow 0$, by virtue of which the only condition that need be imposed on η_x at the wall is $x^{\frac{3}{2}}\eta_x \to 0$. (We anticipate that $\eta_x \to 0$ as $x \downarrow 0$; see §4.) The statement of the problem is completed by the asymptotic condition (1.1), which we recast in the form

$$\eta \sim \mathrm{Re}\{A\,\mathrm{e}^{\mathrm{i}\omega t} \cos(k_0 x + \theta)\} \quad (x \uparrow \infty), \tag{2.7}$$

where

$$A \equiv 2A_1 \mathrm{e}^{-\mathrm{i}\theta}, \quad R \equiv \mathrm{e}^{-2\mathrm{i}\theta}. \tag{2.8 a, b}$$

3. Fourier-integral formulation

We introduce the complex amplitudes $\hat{\phi}$ and $\hat{\eta}$ according to

$$[\phi(x,y,t), \eta(x,t)] = \mathrm{Re}\{[\hat{\phi}(x,y), \hat{\eta}(x)]\,\mathrm{e}^{\mathrm{i}\omega t}\}, \tag{3.1}$$

and satisfy (2.4) and (2.5) by positing

$$[\hat{\phi}(x,y), \hat{\eta}(x)] = \frac{2}{\pi}\int_0^\infty [\Phi(k)\,\mathrm{e}^{ky}, N(k)] \cos kx \, \mathrm{d}k, \tag{3.2 a}$$

where

$$[\Phi(k), N(k)] = \int_0^\infty [\hat{\phi}(x,0), \hat{\eta}(x)] \cos kx \, \mathrm{d}x \tag{3.2 b}$$

are Fourier-cosine transforms. The Fourier integrals in (3.2 *a*) are Cauchy principal values with respect to the singularities of $\Phi(k)$ and $N(k)$ at $k = k_0$, where (2.7) or, equivalently,

$$\hat{\eta} \sim A\cos(k_0 x + \theta) \quad (x \uparrow \infty), \tag{3.3}$$

implies

$$N(k) \sim \tfrac{1}{2}A[\pi\delta(k-k_0)\cos\theta + (k-k_0)^{-1}\sin\theta] \quad (k \to k_0). \tag{3.4}$$

Substituting (3.2 a) into (2.6 a, b), multiplying the resulting equations through by $\cos kx$, integrating over $0 < x < \infty$, introducing

$$m(x, k) \equiv k^{-1}[\exp(ky_{\mathrm{m}}) - 1], \quad q(x) \equiv 1 - (1 + y_{\mathrm{m}}'^2)^{-\frac{3}{2}}, \tag{3.5 a, b}$$

in order to separate out these terms that are directly transformable, and simplifying through integration by parts, we obtain (cf. Miles 1990)

$$\mathrm{i}\omega N - k\Phi = \frac{2k}{\pi}\int_0^\infty \Phi(\kappa)\,\kappa\,\mathrm{d}\kappa \int_0^\infty m(x, \kappa)\sin kx \sin \kappa x\,\mathrm{d}x \tag{3.6 a}$$

and
$$(g + Tk^2)N + \mathrm{i}\omega\Phi = -\frac{2\mathrm{i}\omega}{\pi}\int_0^\infty \Phi(\kappa)\,\kappa\,\mathrm{d}\kappa \int_0^\infty m(x, \kappa)\cos kx \cos \kappa x\,\mathrm{d}x$$
$$+ \frac{2Tk}{\pi}\int_0^\infty N(\kappa)\,\kappa\,\mathrm{d}\kappa \int_0^\infty q(x)\sin kx \sin \kappa x\,\mathrm{d}x. \tag{3.6 b}$$

4. First-order approximation

Guided by (3.4), we pose the solution of (3.6) in the form

$$[N, \Phi] = A\cos\theta\left\{\tfrac{1}{2}\pi\delta(k - k_0)\left[1, \frac{\mathrm{i}\omega}{k}\right] + [N_1, \Phi_1]\right\}, \tag{4.1}$$

where, by hypothesis, N_1 and Φ_1 are $O(y_{\mathrm{m}})$. Substituting (4.1) into (3.6), neglecting the contributions of N_1 and Φ_1 to the right-hand side thereof, approximating $m(x, k_0)$ by $y_{\mathrm{m}}(x)$ (see (3.5 a)), and neglecting the second integral in (3.6 b)†, we obtain the first-order (in y_{m}) approximations

$$\mathrm{i}\omega N_1 - k\Phi_1 = \mathrm{i}\omega k\int_0^\infty y_{\mathrm{m}}(x)\sin kx \sin k_0 x\,\mathrm{d}x \tag{4.2 a}$$

and
$$(g + Tk^2)N_1 + \mathrm{i}\omega\Phi_1 = \omega^2\int_0^\infty y_{\mathrm{m}}(x)\cos kx \cos k_0 x\,\mathrm{d}x. \tag{4.2 b}$$

The elimination of Φ_1 and the identities $T \equiv gl^2$ and $\omega^2 = gk_0(1 + k_0^2 l^2)$ (1.3) then yield

$$N_1 = F(k)\int_0^\infty y_{\mathrm{m}}(x)\cos(k + k_0)x\,\mathrm{d}x, \quad F(k) = \frac{(1 + k_0^2 l^2)\,k_0\,k}{k(1 + k^2 l^2) - k_0(1 + k_0^2 k^2)}. \tag{4.3 a, b}$$

Letting $k \to k_0$ in (4.1) and invoking (3.4), we obtain

$$\tan\theta = 2\lim_{k \to k_0}(k - k_0)N_1(k) \tag{4.4 a}$$
$$= 2k_0^2\left(\frac{1 + k_0^2 l^2}{1 + 3k_0^2 l^2}\right)\int_0^\infty y_{\mathrm{m}}(x)\cos 2k_0 x\,\mathrm{d}x \tag{4.4 b}$$
$$= 2\lambda^2 + O(\lambda^4) \quad (\lambda \equiv k_0 l \downarrow 0), \tag{4.4 c}$$

where (4.4 b) follows from (4.4 a) through (4.3), and (4.4 c) follows through (2.3). Further analysis reveals that the first-order approximation (4.2) implies an error

† The expansion of q in powers of y_{m}' is inadmissible due to the singularity of y_{m}' at $x = 0$, but $0 < q < 1$ for $x = O(l)$ and $q = O(y_{\mathrm{m}}'^2)$ for $x \gg l$, whence it can be shown that the neglect of the second integral in (3.6 b) is consistent with the first-order approximation.

factor of $1+O(\lambda)$, whence the error in (4.4 *c*) is $O(\lambda^3)$, rather than $O(\lambda^4)$; however, it appears that the numerical value of the $O(\lambda)$ term in the error factor is typically smaller than that of the $O(\lambda^2)$ term.

The inverse-Fourier-cosine transform of (4.3 *a*) may be obtained through the partial-fraction expansion of $F(k)$ with respect to the poles at $k = k_0$ and $\pm \mathrm{i}l^{-1}[1+O(\lambda)]$ and entries 1.2(8), 1.2(11), 2.2(11), and 2.2(14) in Erdélyi *et al.* (1954); however, the integrals over the meniscus are intractable, and we therefore consider further only the first approximation to $\hat{\eta}$ at the wall. Combining (3.2 *a*) and (4.1), setting $x = 0$, invoking (4.3 *a, b*) and (2.2), and letting $\lambda \downarrow 0$, we obtain

$$\hat{\eta}(0) = A\cos\theta\left[1+\frac{2}{\pi}\int_0^\infty N_1(k)\,\mathrm{d}k\right] \tag{4.5 a}$$

$$= A\left[1+\frac{2k_0}{\pi}\int_0^\infty \frac{\mathrm{d}k}{1+k^2l^2}\int_0^\infty y_{\mathrm{m}}(x)\cos kx\,\mathrm{d}x+O(\lambda^2)\right] \tag{4.5 b}$$

$$= A\left[1+\lambda\int_0^\infty Y_{\mathrm{m}}(X)\,\mathrm{e}^{-X}\,\mathrm{d}X+O(\lambda^2)\right]. \tag{4.5 c}$$

The rough approximation $Y_{\mathrm{m}} = \exp(-X)$, which satisfies (2.1) for $X \gg 1$ and is normalized to satisfy (2.3), yields $\frac{1}{2}$ for the integral in (4.5 *c*). It follows from (4.3) and $y'_{\mathrm{m}} = O(x^{-\frac{1}{2}})$ as $x \downarrow 0$ that $N_1(k) = O(k^{-\frac{3}{2}})$ as $k \uparrow \infty$, and hence from (3.1) and (4.1) that $\eta_x \to 0$ as $x \downarrow 0$.

5. Standing waves in a channel

Standing waves of complex amplitude $\mathscr{A}$ that are antisymmetric/symmetric (odd/even) with respect to the midplane $x = \frac{1}{2}b$ of the deep channel $0 < x < b$ are described by

$$\hat{\eta}_{\mathrm{o,e}} = \mathscr{A}{\textstyle\genfrac{}{}{0pt}{}{\sin}{\cos}}\,k_0(\tfrac{1}{2}b-x) \quad (l \ll x \leqslant \tfrac{1}{2}b). \tag{5.1}$$

Matching (5.1) to (3.3) in $l \ll x \ll 1/k_0$, we obtain

$$A = (-)^m\mathscr{A}, \quad k_0 b = \begin{pmatrix} 2m+1 \\ 2m \end{pmatrix}\pi - 2\theta. \tag{5.2 a, b}$$

The error in the asymptotic matching between (5.1) and (3.3) corresponds to that in (4.4 *c*), the substitution of which into (5.2 *b*), followed by the invocation of (1.3), yields

$$k_0 b = n\pi - 4\lambda_n^2, \quad \lambda_n = n\pi l/b \quad (n = 1, 2, \ldots), \tag{5.3 a, b}$$

and

$$\omega_n^2 = (g/b)\,[n\pi+(n\pi-4)\,\lambda_n^2]. \tag{5.4}$$

6. Standing waves in a circular cylinder

The effects of the meniscus on standing waves in a cylinder may be calculated by regarding the solution of the reflection problem as a boundary-layer approximation, in which $x = O(l)$ is the normal coordinate and the amplitude A is a slowly varying function of the transverse (to x and y) coordinate, and matching to an appropriate outer solution in $l \ll x \ll 1/k_0$. Note that, in this approximation, the implicit assumption of deep water in the inner approximation requires only that the depth be large compared with the capillary length.

Consider a circular cylinder of radius a and depth h, for which the outer solution has the form (Lamb 1932, §§191 and 257)

$$\hat{\eta} = \mathscr{A}\frac{J_m(kr)}{J_m(ka)}\cos m\psi \quad (a-r \gg l), \tag{6.1}$$

where $k \equiv k_0$, $m = 0, 1, \ldots$ is the azimuthal wavenumber, and r and ψ are polar coordinates. Matching (6.1) to (3.3) in $l \ll x = a-r \ll 1/k$, we obtain

$$\mathscr{A}\cos m\psi\left[1+\frac{J_m'(ka)}{J_m(ka)}(-kx)\right] = A\cos\theta[1-kx\tan\theta], \tag{6.2}$$

which implies

$$A\cos\theta = \mathscr{A}\cos m\psi, \quad J_m'(ka)/J_m(ka) = \tan\theta. \tag{6.3\,a, b}$$

Expanding (6.3 b) about $ka = \kappa$, where κ is a zero of J_m', eliminating J_m'' through Bessel's equation, and invoking $\tan\theta \approx 2\lambda^2$ (4.4 c), we obtain (cf. (5.3))

$$ka = \kappa - 2\left(\frac{\kappa^2}{\kappa^2-m^2}\right)\lambda^2, \quad \lambda = \kappa\frac{l}{a}. \tag{6.4\,a, b}$$

The corresponding approximation to the frequency (now allowing for finite depth) is given by

$$\omega^2 = gk(1+k^2l^2)\tanh kh \tag{6.5\,a}$$

$$= \omega_0^2[1-2\varDelta+O(\lambda^4)], \tag{6.5\,b}$$

where

$$\omega_0^2 \equiv \frac{g\kappa}{a}(1+\lambda^2)\tanh\frac{\kappa h}{a}, \quad \varDelta = \left(\frac{\kappa\lambda^2}{\kappa^2-m^2}\right)\left[1+\frac{2\kappa h/a}{\sinh(2\kappa h/a)}\right]. \tag{6.6\,a, b}$$

$\varDelta$, the meniscus-induced fractional decrement in the frequency, reduces to $0.77\lambda^2$ for the dominant mode ($m = 1$ and $\kappa = 1.84$) in deep ($h > 2a$) water. This decrement exceeds the corresponding increment of $\frac{1}{2}\lambda^2$ associated with the capillary energy of the free surface.

The decrement $-\varDelta\omega_0$ may be comparable with that induced by viscous boundary layers. The Stokes-like boundary layers on the walls are governed by the linear diffusion equation

$$(\nu\partial_x^2 - \mathrm{i}\omega)\,v = 0, \tag{6.7}$$

where ν is the kinematic viscosity, x is the coordinate normal to the wall, and v is the tangential velocity, which must vanish at the wall and match the inviscid velocity outside of the boundary layer. A straightforward boundary-layer analysis† reveals that the viscous shift in frequency is $-\delta_w\omega_0$, where δ_w is the damping ratio associated with boundary-layer dissipation and is given by (Case & Parkinson 1957; Miles 1967)

$$\delta_w = \frac{1}{4a}\left(\frac{2\nu}{\omega_0}\right)^{\frac{1}{2}}\left[\frac{\kappa^2+m^2}{\kappa^2-m^2}+2\kappa\left(1-\frac{h}{a}\right)\operatorname{cosech}\left(2\kappa\frac{h}{a}\right)\right], \tag{6.8}$$

in which $(2\nu/\omega_0)^{\frac{1}{2}}$ is the boundary-layer thickness.

The equality between the frequency decrement and the exponential damping coefficient reflects the fact that the phase of the boundary-layer impedance is $\frac{1}{4}\pi$ and holds for the boundary layer at the free surface only in the limiting case of an

† This analysis ignores the meniscus and implicitly assumes that the first-order effects of the meniscus and the Stokes boundary layers are additive. This assumption is confirmed by the more detailed analysis of Mei & Liu (1973).

inextensible film. In the more general case of a linear, viscoelastic surface for which the relaxation time is small compared with $2\pi/\omega_0$, the counterpart of $(1+\mathrm{i})\delta_w$ is $(\alpha+\mathrm{i}\beta)\,\delta_s$, where $\alpha\delta_s\omega_0$ and $\beta\delta_s\omega_0$ are, respectively, the incremental damping coefficient and decremental frequency induced by the surface film,

$$\alpha+\mathrm{i}\beta = (1+\mathrm{i})\,C = C_r - C_i + \mathrm{i}(C_r + C_i), \quad \delta_s = \frac{1}{4}\frac{\kappa}{a}\left(\frac{2\nu}{\omega_0}\right)^{\frac{1}{2}} \coth\left(\kappa\frac{h}{a}\right), \qquad (6.9\,a, b)$$

and C is given by (3.15) of Miles (1967). The limiting values of C for a clean surface (no film) and a fully contaminated surface (inextensible film) are 0 and 1, respectively. If the film is insoluble and inviscid (perfectly elastic), the damping has a resonant maximum of $\alpha = 2$, with $\beta = 0$ at $C_r = C_i = 1$.

Cocciaro *et al.* (1991) report measurements of frequency and damping coefficient for the dominant mode in a cylinder of radius 5.025 cm filled to a depth of 7.8 cm with a fluid ('octane') that appeared to wet the wall with zero ($\leqslant 2°$) contact angle. They compare their measured frequencies with $\omega = \omega_0(1-2\delta_w)$, which they attribute to Mei & Liu (1973), but the factor of 2 in their (1.5) is a misprint and should be deleted. They state that the frequency given by Mei & Liu's result for the dominant mode in their circular cylinder is $\nu_1 \equiv \omega/2\pi = 3.000$ Hz; however, using their data ($a = 5.025$ cm, $h = 7.8$ cm, $\kappa = 1.8412$, $T = 21.80$ dynes/cm, $\rho = 0.702$ gm/cm^3, $\nu = 0.00772$ cm^2/s), and $g = 980.5$ cm/s^2, I obtain (B. Cocciaro, private communication, agrees with this revised calculation of ν_0)

$$\nu_0 \equiv \omega_0/2\pi = 3.0131 \text{ Hz}, \quad \delta_w = 2.60\times10^{-3}, \quad \Delta = 3.39\times10^{-3}, \qquad (6.10\,a\text{–}c)$$

$$\nu_0(1-\delta_w) = 3.005 \text{ Hz}, \quad \nu_0(1-\delta_w-\Delta) = 2.995 \text{ Hz}. \qquad (6.10\,d, e)$$

This last result is close to their measured frequency of 2.997 Hz in the limit of zero amplitude (see their figure 4*b*). However, their corresponding damping coefficient is $\pi\gamma = 0.107$ s^{-1}, whereas the theoretical result for a clean surface is $\delta_w\omega_0 = 0.049$ s^{-1}. It is possible that this discrepancy is due to surface contamination (the theoretical, incremental damping coefficient for a fully contaminated surface is $\delta_s\omega_0 = 0.050$ s^{-1}, which also would imply a corresponding reduction of 8 mHz in the theoretical frequency), but Cocciaro *et al.* (1991) regard this as unlikely and suggest that 'the effect of the draining film at the vertical wall could be dominant for the damping. (Cocciaro *et al.* 1991 also invoke my (Miles 1967) model of capillary-hysteresis to obtain finite-amplitude damping in qualitative agreement with their observations, but this mechanism is absent for zero amplitude.)

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Appendix. Viscous dissipation

The constraint of zero contact angle implies that no work is done by the capillary force at the contact line. The viscous dissipation rate (per unit width in the boundary layer at the wall) is given by (Miles 1967)

$$D = \tfrac{1}{2}\rho(2\nu\omega)^{\frac{1}{2}} \int_{-\infty}^{0} |\hat{v}|^2\, \mathrm{d}y, \qquad (\mathrm{A}\,1)$$

where ρ is the density, ν is the kinematic viscosity, and

$$\hat{v} = \hat{\phi}_y|_{x=0} = \frac{2}{\pi}\int_0^\infty \Phi(k)\,\mathrm{e}^{ky}k\,\mathrm{d}k \tag{A 2}$$

is the complex amplitude of the velocity at the wall. The first-order approximation, obtained from (4.1) and (4.2), is

$$\hat{v} = \mathrm{i}\omega A\cos\theta\left[\mathrm{e}^{k_0 y}+\frac{2}{\pi}(\mathrm{i}\omega)^{-1}\int_0^\infty \Phi_1(k)\,\mathrm{e}^{ky}k\,\mathrm{d}k\right]. \tag{A 3}$$

Substituting (A 3) into (A 1), and neglecting the second-order term, we obtain

$$\int_{-\infty}^0 |\hat{v}|^2\mathrm{d}y = \frac{\omega^2|A|^2\cos^2\theta}{2k_0}\left[1+\frac{8k_0}{\pi\mathrm{i}\omega}\int_0^\infty\frac{k\Phi_1(k)\,\mathrm{d}k}{k+k_0}\right] \tag{A 4a}$$

$$= \tfrac{1}{2}\omega^2|A|^2k_0^{-1}[1+O(\lambda^2)], \tag{A 4b}$$

where (A 4*b*) follows from (A 4*a*) through (4.2)–(4.4).

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